

Convergence to equilibrium of global weak solutions for a Cahn-Hilliard-Navier-Stokes vesicle model

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Abstract

A model about the dynamic of vesicle membranes in incompressible viscous fluids is introduced. The system consists of the Navier-Stokes equations with an extra stress depending on the membrane, coupled with a Cahn-Hilliard phase-field equation in 3D domains. This problem has a time dissipative free energy which leads, in particular, to the existence of global in time weak solutions. The asymptotic behavior of these solutions is analyzed. Firstly, we prove that the ω -limit set are critical points of the energy. Afterwards, by using a modified Łojasiewicz-Simon's inequality, we demonstrate the convergence of the whole trajectory to a single equilibrium. Finally, the asymptotic convergence of the phase field is improved. It is remarkable that all these arguments may be done without using strong regularity for the velocity and pressure variables.

Keywords: Vesicle membranes, Navier-Stokes equations, Cahn-Hilliard equation, energy dissipation, convergence to equilibrium, Łojasiewicz-Simon's inequalities.

1 Introduction

In this paper, we consider a model for the dynamic of vesicle membranes in incompressible viscous fluids, developed from the geometry of the membranes. This type of models was introduced by Helfrich [7]. The system consists of the Navier-Stokes equations with an extra stress depending on the membrane, coupled with a Cahn-Hilliard phase-field equation.

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The membranes are formed by lipid bilayers. Under appropriate conditions, they withdraw into itself forming a sort of bag, named vesicle. The equilibrium configurations of vesicle membranes can be obtained minimizing the Helfrich bending elastic energy, under fixed surface area and volume constraints.

Numerous studies have been devoted to this type of models and a detailed description of they can be seen in [4] and references therein. A phase function is also used to model the vesicle membrane as a diffuse interface in subsequent papers. In [9] and [12], a coupled Allen-Cahn and Navier-Stokes problem is studied approaching both constraints, area and volume, in a penalized manner. In [1], a Cahn-Hilliard phase-field model is introduced, without taking into account the vesicle-fluid interaction. Now a Cahn-Hilliard-Navier-Stokes model will be considered. Since the volume constraint is implicitly satisfied in the Cahn-Hilliard equation, now only the surface area constraint must be approximated via penalization.

We will prove that the resulting problem is thermodynamically consistent, because there exists a free energy (kinetic plus bending plus penalized one) which is dissipative in time along the trajectories. This fact is used to prove the existence of global in time weak solutions.

On the other hand, the asymptotic behavior of the solutions is analyzed following the way of [6], [10], [2]. We prove that the ω -limit set for weak solutions are critical points of the dissipative energy. After that, by using a modified Łojasiewicz-Simon's inequality we demonstrate the convergence of the whole trajectory to a single equilibrium. We consider of remarkable interest in this paper two facts: The study of the new model Navier-Stokes-Cahn-Hilliard modeling vesicles and the identification of a unique equilibrium point in the context of weak solutions.

The current paper is organized as follows. We explain the model in Section 2 and give some preliminary results in Section 3. In Section 4, an energy equality and some global weak estimates are obtained that let us to prove the existence of weak solutions. Section 5 is devoted to the study of convergence at infinite time for global weak solutions. We prove that the ω -limit set are critical points. After that, by using a modified Łojasiewicz-Simon's inequality, we demonstrate that each trajectory converges to a single equilibrium. In section 6, some global in time strong estimates for the phase are obtained, which let us to improve the convergence of the phase trajectory in a higher order space.

2 The model

We will analyze the case where the bending energy E_b is given by a simplified elastic Willmore energy plus a penalization of the surface area constraint [5]:

$$E_b(\phi) = \frac{1}{2\varepsilon} \int_{\Omega} (-\varepsilon \Delta \phi + \frac{1}{\varepsilon} F'(\phi))^2 dx + \frac{1}{2} M (A(\phi) - \alpha)^2 \quad (1)$$

where $F'(\phi) = (|\phi|^2 - 1)\phi$ for each $\phi \in \mathbb{R}$, being $F(\phi) = \frac{1}{4}(\phi^2 - 1)^2$ the Ginzburg-Landau potential, $M > 0$ is a penalization constant, ε is related to the interface width, and

$$A(\phi) = \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla \phi|^2 + \frac{1}{\varepsilon} F'(\phi) \right) dx,$$

approaches the surface area.

Remark 1 *The same results of this paper may be obtained replacing the surface area $A(\phi)$ only by the first term $A(\phi) = \int_{\Omega} \frac{\varepsilon}{2} |\nabla \phi|^2 dx$ as in [1].*

By introducing the chemical potential

$$w := \frac{\delta E_b(\phi)}{\delta \phi}$$

we will introduce the following Navier-Stokes-Cahn-Hilliard equations in $\Omega \times (0, +\infty)$:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} - \lambda w \nabla \phi + \nabla q = 0, \quad (2)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (3)$$

$$\partial_t \phi + \mathbf{u} \cdot \nabla \phi - \gamma \Delta w = 0. \quad (4)$$

The coefficients $\nu > 0$, $\lambda > 0$ and $\gamma > 0$ depend on viscosity, elasticity and mobility, respectively. The system (2)-(4) is completed with the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \partial_n \phi|_{\partial\Omega} = 0, \quad \partial_n \Delta \phi|_{\partial\Omega} = 0, \quad \partial_n w|_{\partial\Omega} = 0, \quad (5)$$

and the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \phi(0) = \phi_0 \quad \text{in } \Omega. \quad (6)$$

For compatibility, we will assume $\mathbf{u}_0|_{\partial\Omega} = 0$ with $\nabla \cdot \mathbf{u}_0 = 0$ and $\partial_n \phi_0|_{\partial\Omega} = 0$.

By integrating the w -equation (4), using the free-divergence $\nabla \cdot \mathbf{u} = 0$, the non-slip condition $\mathbf{u}|_{\partial\Omega} = 0$, and the last boundary condition $\partial_n w|_{\partial\Omega} = 0$, it is easy to deduce

$$\frac{d}{dt} \int_{\Omega} \phi(x, t) dx = 0,$$

hence one has the conservative character of ϕ in Ω , because the total volume is conserved:

$$V(t) := \int_{\Omega} \phi(x, t) dx = \int_{\Omega} \phi_0(x) dx := V_0 \in \mathbb{R}.$$

On the one hand, for all $\phi, \bar{\phi} \in H^1$,

$$\left\langle \frac{\delta A(\phi)}{\delta \phi}, \bar{\phi} \right\rangle = \int_{\Omega} \varepsilon \nabla \phi \cdot \nabla \bar{\phi} + \frac{1}{\varepsilon} F'(\phi) \bar{\phi},$$

hence integrating by parts, if $\phi \in H^2$ and $\partial_n \phi|_{\partial\Omega} = 0$, one can identify

$$\mu := \frac{\delta A(\phi)}{\delta \phi} = -\varepsilon \Delta \phi + \frac{1}{\varepsilon} F'(\phi),$$

Note that, by using the boundary conditions for ϕ given in (5), it also holds $\nabla \mu \cdot \mathbf{n}|_{\partial\Omega} = 0$.

On the other hand, for all $\phi, \bar{\phi} \in H^2$,

$$\left\langle \frac{\delta E_b(\phi)}{\delta \phi}, \bar{\phi} \right\rangle = \frac{1}{\varepsilon} \int_{\Omega} \mu (-\varepsilon \Delta \bar{\phi} + \frac{1}{\varepsilon} F'(\phi) \bar{\phi}) + M(A(\phi) - \alpha) \left(\int_{\Omega} \varepsilon \nabla \phi \cdot \nabla \bar{\phi} + \frac{1}{\varepsilon} F'(\phi) \bar{\phi} \right),$$

hence, after some integrations by parts, using $\nabla \phi \cdot \mathbf{n}|_{\partial\Omega} = 0$, $\nabla \mu \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $\nabla \bar{\phi} \cdot \mathbf{n}|_{\partial\Omega} = 0$, one can identify

$$w := \frac{\delta E_b(\phi)}{\delta \phi} = -\Delta \mu + \frac{1}{\varepsilon^2} \mu F'(\phi) + M(A(\phi) - \alpha) \mu$$

Remark 2 The variational derivatives $w := \frac{\delta E_b(\phi)}{\delta \phi}$ and $\mu := \frac{\delta A(\phi)}{\delta \phi}$ have been identified as $L^2(\Omega)$ -functions via the $L^2(\Omega)$ scalar product.

In particular, we can decompose

$$w = \varepsilon \Delta^2 \phi + G(\phi) \tag{7}$$

with

$$G(\phi) := -\frac{1}{\varepsilon} \Delta F'(\phi) + \frac{1}{\varepsilon^2} \mu F'(\phi) + M(A(\phi) - \alpha) \mu. \tag{8}$$

Since $\partial_n \phi|_{\partial\Omega} = 0$, in particular $\partial_n F'(\phi)|_{\partial\Omega} = 0$, hence

$$\int_{\Omega} -\Delta F'(\phi) \, dx = \int_{\partial\Omega} -F''(\phi) (\nabla \phi \cdot \mathbf{n}) \, dx = 0.$$

Therefore, integrating (7) and (8),

$$\int_{\Omega} w \, dx = \int_{\Omega} G(\phi) \, dx = \frac{1}{\varepsilon^2} \int_{\Omega} \mu F'(\phi) \, dx + M(A(\phi) - \alpha) \int_{\Omega} \mu \, dx. \tag{9}$$

By using $w = \varepsilon \Delta^2 \phi + G(\phi)$ as auxiliary variable, we can rewrite the problem (2)-(4) as

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} - \lambda w \nabla \phi + \nabla q = 0, \tag{10}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{11}$$

$$\partial_t \phi + \mathbf{u} \cdot \nabla \phi - \gamma \Delta w = 0, \tag{12}$$

$$\varepsilon \Delta^2 \phi + G(\phi) - w = 0, \tag{13}$$

With the aim to use the conservation of the problem, if we denote $m_0 = \langle \phi_0 \rangle := \frac{1}{|\Omega|} \int_{\Omega} \phi_0(x) dx$, we introduce the following mean-value variables for the phase-field problem:

$$\psi(x, t) := \phi(x, t) - m_0 \quad \text{and} \quad z := w - \langle G(\phi) \rangle.$$

From (9), $\langle G(\phi) \rangle = \frac{1}{\varepsilon^2} \langle \mu F'(\phi) \rangle + M(A(\phi) - \alpha) \langle \mu \rangle$. Reciprocally, given (ψ, z) we can recover (ψ, w) as $\psi = \psi + m_0$ and $w = z + \langle G(\phi) \rangle$.

By rewriting the equations (10)-(13) in these new variables (ψ, z) we arrive at the problem

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} - \lambda z \nabla \psi + \nabla \tilde{q} = 0, \quad (14)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (15)$$

$$\partial_t \psi + \mathbf{u} \cdot \nabla \psi - \gamma \Delta z = 0, \quad (16)$$

$$\varepsilon \Delta^2 \psi + \overline{G}(\psi) - z = 0, \quad (17)$$

where

$$\tilde{q} = q + \lambda \langle G(\phi) \rangle \psi$$

and

$$\overline{G}(\psi) := G(\psi + m_0) - \langle G(\psi + m_0) \rangle,$$

completed with the boundary and initial conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \partial_n \psi|_{\partial\Omega} = 0, \quad \partial_n \Delta \psi|_{\partial\Omega} = 0, \quad \partial_n z|_{\partial\Omega} = 0, \quad (18)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \psi(0) = \psi_0 := \phi_0 - \langle \phi_0 \rangle \quad \text{in } \Omega. \quad (19)$$

Observe that, $\int_{\Omega} \psi dx = 0$ and $\int_{\Omega} z dx = 0$.

Finally, by denoting

$$\overline{E}_b(\psi) = E_b(\psi + m_0) \quad (20)$$

then,

$$z = \frac{\delta \overline{E}_b(\psi)}{\delta \psi} = \varepsilon \Delta^2 \psi + \overline{G}(\psi), \quad (21)$$

where the identification (21) has been done via the $L_*^2(\Omega)$ -scalar product.

3 Some preliminary results

Let us to introduce the following notations:

- In general, the notation will be abridged. We set $L^p = L^p(\Omega)$, $p \geq 1$, $H_0^1 = H_0^1(\Omega)$, etc. If $X = X(\Omega)$ is a space of functions defined in the open set Ω , we denote by $L^p(0, T; X)$ the Banach space $L^p(0, T; X(\Omega))$. Also, boldface letters will be used for vectorial spaces, for instance $\mathbf{L}^2 = L^2(\Omega)^N$.

- The L^p -norm is denoted by $|\cdot|_p$, $1 \leq p \leq \infty$, the H^m -norm by $\|\cdot\|_m$ (in particular $|\cdot|_2 = \|\cdot\|_0$). The inner product of $L^2(\Omega)$ is denoted by (\cdot, \cdot) . The boundary $H^s(\partial\Omega)$ -norm is denoted by $\|\cdot\|_{s;\partial\Omega}$.
- We set \mathcal{V} the space formed by all fields $\mathbf{u} \in C_0^\infty(\Omega)^N$ satisfying $\nabla \cdot \mathbf{u} = 0$. We denote \mathbf{H} (respectively \mathbf{V}) the closure of \mathcal{V} in \mathbf{L}^2 (respectively \mathbf{H}^1). \mathbf{H} and \mathbf{V} are Hilbert spaces for the norms $|\cdot|_2$ and $\|\cdot\|_1$, respectively. Furthermore,

$$\mathbf{H} = \{\mathbf{u} \in \mathbf{L}^2; \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \quad \mathbf{V} = \{\mathbf{u} \in \mathbf{H}^1; \nabla \cdot \mathbf{u} = 0, \mathbf{u} = 0 \text{ on } \partial\Omega\}$$

- From now on, $C > 0$ will denote different constants, depending only on the fixed data of the problem.

Let us consider the following mean-value spaces:

$$\begin{aligned} L_*^2 &= \left\{ w \in L^2(\Omega); \int_{\Omega} w = 0 \right\}, \\ H_*^k &= \left\{ w \in H^1(\Omega); \int_{\Omega} w = 0 \right\} \quad k \geq 1, \\ H_1^2 &= \{w \in H_*^2(\Omega); \partial_n w|_{\partial\Omega} = 0\} \\ H_2^k &= \{w \in H_*^k; \partial_n w|_{\partial\Omega} = 0, \partial_n \Delta w|_{\partial\Omega} = 0\} \quad k = 3, 4, 5, 6. \end{aligned}$$

We will assume Ω regular enough to use the regularity of the two following elliptic Laplacian-Neuman and Bilaplacian-Dirichlet-Neumann problems, respectively:

$$\begin{cases} -\Delta z = f & \text{in } \Omega \\ \partial_n z|_{\partial\Omega} = 0, \quad \int_{\Omega} z \, dx = 0, \end{cases} \quad \begin{cases} \Delta^2 \psi = f & \text{in } \Omega \\ \partial_n \psi|_{\partial\Omega} = 0, \quad \partial_n \Delta \psi|_{\partial\Omega} = 0, \\ \int_{\Omega} \psi \, dx = 0 \end{cases}$$

where $f \in L_*^2(\Omega)$. From the H^2 -regularity of the first problem, the following norms are equivalents:

$$\|z\|_2 \approx |\Delta z|_2 \quad \text{in } H_1^2, \quad (22)$$

and from the H^4 , H^5 and H^6 -regularity of the second problem,

$$\|\psi\|_4 \approx |\Delta^2 \psi|_2 \quad \text{in } H_2^4, \quad \|\psi\|_5 \approx \|\Delta^2 \psi\|_1 \quad \text{in } H_2^5, \quad \|\psi\|_6 \approx \|\Delta^2 \psi\|_2 \quad \text{in } H_2^6. \quad (23)$$

Now we study the elliptic problem relating μ to ψ :

Lemma 3 *Given μ , we consider ψ as the solution of problem*

$$\begin{cases} -\varepsilon \Delta \psi + \frac{1}{\varepsilon} F'(\psi + m_0) = \mu & \text{in } \Omega \\ \partial_n \psi|_{\partial\Omega} = 0, \quad \int_{\Omega} \psi \, dx = 0. \end{cases}$$

Then,

$$\|\psi\|_1 \leq C(1 + |\mu|_2), \quad (24)$$

$$\|\psi\|_2 \leq C(1 + |\mu|_2 + \|\psi\|_1^3), \quad (25)$$

Proof. Firstly, by taking $\psi + m_0$ as test function, we obtain

$$\varepsilon |\nabla \psi|_2^2 + \frac{1}{\varepsilon} |\psi + m_0|_4^4 = (\mu, \psi + m_0) + \frac{1}{\varepsilon} |\psi + m_0|_2^2.$$

By using Young, Holder inequalities and the Poincaré inequality for mean-value functions we obtain:

$$C\varepsilon \|\psi\|_1^2 + \frac{1}{\varepsilon} |\psi + m_0|_4^4 \leq \frac{1}{2} |\mu|_2^2 + \left(\frac{1}{2} + \frac{1}{\varepsilon}\right) |\psi + m_0|_2^2 \leq \frac{1}{2} |\mu|_2^2 + \frac{1}{2\varepsilon} |\psi + m_0|_4^4 + C,$$

hence,

$$\|\psi\|_1^2 \leq C |\mu|_2^2 + C$$

and (24) holds. Secondly, from the regularity of the problem (bootstrap's argument)

$$\begin{cases} -\varepsilon \Delta \psi = \mu - \frac{1}{\varepsilon} F'(\psi + m_0) & \text{in } \Omega \\ \partial_n \psi|_{\partial\Omega} = 0, \int_{\Omega} \psi \, dx = 0, \end{cases}$$

we obtain that $\|\psi\|_2 \leq C(|\mu|_2 + \frac{1}{\varepsilon} |F'(\psi + m_0)|_2)$. From the definition of F' ,

$$|F'(\psi + m_0)|_2 \leq C(|\psi + m_0|_6^3 + |\psi + m_0|_2) \leq C(\|\psi + m_0\|_1^3 + |\psi + m_0|_2) \leq C(1 + \|\psi\|_1^3).$$

Therefore, $\|\psi\|_2 \leq C(1 + |\mu|_2 + \|\psi\|_1^3)$ and (25) holds. ■

We introduce a modified Łojasiewicz-Simon's inequality.

Lemma 4 (Łojasiewicz-Simon inequality) *Let \mathcal{S} be the following set of equilibrium points related to the bending energy $\overline{E}_b(\psi)$ given in (20):*

$$\mathcal{S} = \{\psi \in H_2^4(\Omega) : \varepsilon \Delta^2 \psi + \overline{G}(\psi) = 0 \text{ a.e in } \Omega\}.$$

Let $\overline{\psi} \in \mathcal{S}$ and $K > 0$ fixed. Then, for any two sufficiently small constants β and δ , there exists $C > 0$ and $\theta \in (0, 1/2)$ (depending on $\overline{\psi}$, β and δ), such that for all $\psi \in H_2^4$ with $\|\psi\|_3 \leq K$, $\|\psi - \overline{\psi}\|_1 \leq \beta$ and $|\overline{E}_b(\psi) - \overline{E}_b(\overline{\psi})| \leq \delta$, it holds

$$|\overline{E}_b(\psi) - \overline{E}_b(\overline{\psi})|^{1-\theta} \leq C |z|_2 \quad (26)$$

where $z = z(\psi) := \varepsilon \Delta^2 \psi + \overline{G}(\psi)$.

Proof.

Step 1: For β_1 small enough, there exists $C > 0$ and $\theta \in (0, 1/2)$ (depending on $\bar{\psi}$ and β_1) such that for all $\psi \in H_2^4$ with $\|\psi - \bar{\psi}\|_4 \leq \beta_1$, then (62) holds.

The proof of this step is based in the generic result Theorem 4.2 of [8] for the following spaces and operators (with the same notation that in [8]):

$$\begin{aligned} H &\equiv \tilde{X} = L_*^2(\Omega), & X &\equiv H_A = H_2^4, \\ A &= \Delta^2 : \psi \in X \mapsto A\psi = \Delta^2\psi \in H \text{ and } \langle \psi, \xi \rangle_A = (\Delta^2\psi, \Delta^2\xi)_{L^2} \quad \forall \xi, \psi \in D(A), \\ \mathcal{E} &: \psi \in X \mapsto \mathcal{E}(\psi) = \bar{E}_b(\psi) \in \mathbb{R}, \quad \text{such that} \\ \bar{E}_b(\psi) &= \bar{E}_b(\psi + m_0) = \frac{1}{2\varepsilon} \int_{\Omega} (-\varepsilon\Delta\psi + \frac{1}{\varepsilon}F'(\psi + m_0))^2 dx + \frac{1}{2}M(A(\psi + m_0) - \alpha)^2 \\ \mathcal{M} &= \mathcal{E}' : \psi \in X \mapsto \mathcal{M}(\psi) = \varepsilon\Delta^2\psi + \mathcal{R}(\psi) - \langle \mathcal{R}(\psi) \rangle \in H, \end{aligned}$$

where $\mathcal{R}(\psi) = G(\psi + m_0)$ hence $\bar{G}(\psi) = \mathcal{R}(\psi) - \langle \mathcal{R}(\psi) \rangle$. Finally, $\mathcal{M}_1(\psi) = \mathcal{M}'(\psi)$, where

$$\mathcal{M}'(\psi) : \psi \in X \mapsto \mathcal{M}'(\psi)(\xi) := \varepsilon\Delta^2\xi + \mathcal{R}'(\psi)(\xi) - \langle \mathcal{R}'(\psi)(\xi) \rangle \in H,$$

with

$$\begin{aligned} \mathcal{R}'(\psi)(\xi) &= -\frac{2}{\varepsilon}[F'''(\psi + m_0)\nabla\psi \cdot \nabla\xi + 3|\nabla\psi|^2\xi + F''(\psi + m_0)\Delta\xi + F'''(\psi + m_0)(\Delta\psi)\xi] \\ &\quad + \frac{1}{\varepsilon^3}[(F''(\psi + m_0))^2 + F'''(\psi + m_0)F'(\psi + m_0)]\xi \\ &\quad + M(-\varepsilon\Delta\psi + \frac{1}{\varepsilon}F'(\psi + m_0)) \int_{\Omega} \nabla\psi \cdot \nabla\xi dx + M(A(\psi + m_0) - \alpha)(-\varepsilon\Delta\xi + \frac{1}{\varepsilon}F''(\psi + m_0)\xi). \end{aligned}$$

Note that $\mathcal{M}'(\psi)$ is a Fredholm operator of index zero, because $\mathcal{M}'(\psi)$ is the sum of the invertible operator A and a compact operator.

Moreover, the map $\mathcal{R} : \psi \in X \mapsto \mathcal{M}'(\psi)A^{-1} \in \mathcal{L}(H)$ is well-posed because $A^{-1} \in \mathcal{L}(H; X)$ and $\mathcal{M}'(\psi) \in \mathcal{L}(X; H)$. It remains to be proved that \mathcal{R} is (sequentially) continuous. Let $\psi_n \rightarrow \psi$ in X as $n \rightarrow \infty$. Then,

$$\begin{aligned} \|\mathcal{R}(\psi_n) - \mathcal{R}(\psi)\|_{\mathcal{L}(H)} &= \|\mathcal{M}'(\psi_n)A^{-1} - \mathcal{M}'(\psi)A^{-1}\|_{\mathcal{L}(H)} \\ &\leq \|\mathcal{M}'(\psi_n) - \mathcal{M}'(\psi)\|_{\mathcal{L}(X; H)}\|A^{-1}\|_{\mathcal{L}(H; X)} \end{aligned}$$

Since

$$\begin{aligned}
\mathcal{R}'(\psi_n)(\xi) - \mathcal{R}'(\psi)(\xi) = & -\frac{2}{\varepsilon}[F'''(\psi_n + m_0)\nabla(\psi_n - \psi) + (F'''(\psi_n + m_0) - F'''(\psi + m_0))\nabla\psi]\nabla\xi \\
& -\frac{2}{\varepsilon}[3(\nabla\psi_n + \nabla\psi)(\nabla\psi_n - \nabla\psi)\xi + (F''(\psi_n + m_0) - F''(\psi + m_0))\Delta\xi] \\
& -\frac{12}{\varepsilon}[(\psi_n + m_0)\Delta(\psi_n - \psi) + (\psi_n - \psi)\Delta\psi]\xi + \frac{1}{\varepsilon^2}[F''(\psi + m_0)3((\psi_n + m_0)^2 - (\psi + m_0)^2) \\
& + (F''(\psi_n + m_0) - F''(\psi + m_0))(3(\psi_n + m_0) - 1) + 6(\psi_n + m_0)(F'(\psi_n + m_0) - F'(\psi + m_0)) \\
& + 6(\psi_n - \psi)F'(\psi + m_0)]\xi + M[(-\varepsilon\Delta\psi_n + \frac{1}{\varepsilon}F'(\psi + m_0)) \int_{\Omega} \nabla(\psi_n - \psi)\nabla\xi \, dx \\
& (-\varepsilon\Delta(\psi_n - \psi) + \frac{1}{\varepsilon}(F'(\psi_n + m_0) - F'(\psi + m_0))) \int_{\Omega} \nabla\psi\nabla\xi \, dx] \\
& + M(-\varepsilon\Delta\xi + \frac{1}{\varepsilon}F''(\psi + m_0)\xi) \int_{\Omega} \frac{\varepsilon}{2}(\nabla\psi_n - \nabla\psi)(\nabla\psi_n + \nabla\psi) \, dx \\
& + M(-\varepsilon\Delta\xi + \frac{1}{\varepsilon}F''(\psi + m_0)\xi) \int_{\Omega} \frac{1}{4\varepsilon}((\psi_n + m_0)^2 + (\psi + m_0)^2 - 2)(\psi_n + \psi + 2m_0)(\psi_n - \psi) \, dx \\
& + \frac{1}{\varepsilon}(A(\psi + m_0) - \alpha)(F''(\psi_n + m_0) - F''(\psi + m_0))
\end{aligned}$$

and the differences $F'(\psi_n + m_0) - F'(\psi + m_0)$, $F''(\psi_n + m_0) - F''(\psi + m_0)$ and $F'''(\psi_n + m_0) - F'''(\psi + m_0)$ contain the factor $\psi_n - \psi$, then

$$\begin{aligned}
\|\mathcal{M}'(\psi_n) - \mathcal{M}'(\psi)\|_{\mathcal{L}(X;H)} &= \sup_{\xi \in X \setminus \{0\}} \frac{\|\mathcal{M}'(\psi_n)(\xi) - \mathcal{M}'(\psi)(\xi)\|_H}{\|\xi\|_X} \\
&= \sup_{\xi \in X \setminus \{0\}} \frac{|\overline{G}'(\psi_n)(\xi) - \overline{G}'(\psi)(\xi)|_2}{\|\xi\|_4} \leq C(\|\psi_n - \psi\|_{L^2} + \|\psi_n - \psi\|_{H^1} + \|\psi_n - \psi\|_{H^2}).
\end{aligned}$$

Therefore, $\|\mathcal{M}'(\psi_n) - \mathcal{M}'(\psi)\|_{\mathcal{L}(X;H)} \rightarrow 0$ as $n \rightarrow \infty$ if $\psi_n \rightarrow \psi$ in H^4 (even if $\psi_n \rightarrow \psi$ in H^2), hence the continuity of the operator \mathcal{R} is deduced.

For any $\psi \in H_2^4(\Omega)$ satisfying $\|\psi - \overline{\psi}\|_4 \leq \beta_1$, owing to Theorem 4.2 of [8], there exists $C > 0$ and $\theta \in (0, 1/2)$ such that:

$$|\overline{E}_b(\psi) - \overline{E}_b(\overline{\psi})|^{1-\theta} \leq C \|\mathcal{E}'(\psi)\|_H = C |z(\psi)|_2$$

and (62) holds.

Step 2: (Relaxing the local approximation $\|\psi - \overline{\psi}\|_4 \leq \beta_1$ by $\|\psi\|_3 \leq K$, $\|\psi - \overline{\psi}\|_1 \leq \beta$ and $|\overline{E}_b(\psi) - \overline{E}_b(\overline{\psi})| \leq \delta$). There exists $C > 0$ and $\theta \in (0, 1/2)$ (depending on $\overline{\psi}$, β , K and δ) such that if $\psi \in H_2^4(\Omega)$ with $\|\psi\|_3 \leq K$, $\|\psi - \overline{\psi}\|_1 \leq \beta$ and $|\overline{E}_b(\psi) - \overline{E}_b(\overline{\psi})| \leq \delta$, then (62) holds.

In this step, we follow Lemma 4.4 of [11] but imposing the “proximity” condition between ψ and $\overline{\psi}$ only in the H^1 -norm instead of the H^2 -norm as in [11] by using the convergence of the energies.

Firstly, there is a constant $M > 0$, such that

$$\|\psi - \overline{\psi}\|_4 \leq M|\Delta^2(\psi - \overline{\psi})|_2$$

(here, $\int \psi \, dx = \int \bar{\psi} \, dx$ has been used), hence,

$$|z(\psi)|_2 \geq |\varepsilon \Delta^2(\psi - \bar{\psi})| - |\bar{G}(\psi) - \bar{G}(\bar{\psi})|_2 \geq \frac{\varepsilon}{M} \|\psi - \bar{\psi}\|_4 - |\bar{G}(\psi) - \bar{G}(\bar{\psi})|_2. \quad (27)$$

Secondly, we are going to bound $|\bar{G}(\psi) - \bar{G}(\bar{\psi})|_2$. For this,

$$\begin{aligned} \bar{G}(\psi) - \bar{G}(\bar{\psi}) &= -k[F'''(\psi + m_0)\nabla(\psi + \bar{\psi})\nabla\psi + 6\psi(\nabla\bar{\psi})^2] \\ &\quad - 2k[F''(\psi + m_0)\Delta\psi + 3(\psi + \bar{\psi} + 2m_0)\psi\Delta\bar{\psi}] \\ &\quad + \frac{k}{\varepsilon^2}[F''(\psi + m_0)\psi((\psi + m_0)^2 + (\psi + m_0)(\bar{\psi} + m_0) + (\bar{\psi} + m_0)^2 - 1) \\ &\quad + 3\psi(\psi + \bar{\psi} + 2m_0)F'(\bar{\psi} + m_0)] \\ &\quad + \frac{M\varepsilon^2}{2}(-\Delta\psi + \frac{1}{\varepsilon}F'(\psi + m_0)) \int_{\Omega} \nabla(\psi + \bar{\psi})\nabla\psi \, dx \\ &\quad + M\varepsilon(B(\bar{\psi}) - \beta)(-\Delta\psi + \frac{1}{\varepsilon}(F'(\psi + m_0) - F'(\bar{\psi} + m_0))). \end{aligned}$$

Then, there is a constant $C_1 = C_1(K) > 0$ such that

$$|\bar{G}(\psi) - \bar{G}(\bar{\psi})|_2 \leq C_1(K) \|\psi - \bar{\psi}\|_2.$$

In particular, interpolating H_1^2 between H_*^1 and H_2^4 ,

$$|\bar{G}(\psi) - \bar{G}(\bar{\psi})|_2 \leq C_1(K) \|\psi - \bar{\psi}\|_1^{1/2} \|\psi - \bar{\psi}\|_4^{1/2} \leq \frac{\varepsilon}{2M} \|\psi - \bar{\psi}\|_4 + \frac{M}{2\varepsilon} C_1(K)^2 \|\psi - \bar{\psi}\|_1.$$

Let $\beta_1 > 0$, $\theta \in (0, 1/2)$ given in Step 1, by choosing $\delta > 0$ and $\beta > 0$, both sufficiently small, such that

$$\frac{M}{2\varepsilon} C_1(K)^2 \beta \leq \frac{\varepsilon\beta_1}{4M} \quad \text{and} \quad \delta^{1-\theta} \leq \frac{\varepsilon\beta_1}{4M},$$

then, for any $\psi \in H_2^4(\Omega)$ satisfying $\|\psi - \bar{\psi}\|_2 \leq \beta$ and $|\bar{E}_b(\psi) - \bar{E}_b(\bar{\psi})| \leq \delta$ we have

$$|\bar{G}(\psi) - \bar{G}(\bar{\psi})|_2 \leq \frac{\varepsilon}{2M} \|\psi - \bar{\psi}\|_4 + \frac{\varepsilon\beta_1}{4M} \quad (28)$$

and

$$|\bar{E}_b(\psi) - \bar{E}_b(\bar{\psi})|^{1-\theta} \leq \frac{\varepsilon\beta_1}{4M}. \quad (29)$$

There are only two possibilities: either $\|\psi - \bar{\psi}\|_4 < \beta_1$ and then (62) holds by using Step 1; or $\|\psi - \bar{\psi}\|_4 > \beta_1$. In this latter case, from (27) and (28)

$$\begin{aligned} |z(\psi)|_2 &\geq \frac{\varepsilon}{2M} \|\psi - \bar{\psi}\|_4 - \frac{M}{2\varepsilon} C_1(K)^2 \|\psi - \bar{\psi}\|_1 \\ &> \frac{\varepsilon\beta_1}{2M} - \frac{\varepsilon\beta_1}{4M} = \frac{\varepsilon\beta_1}{4M}. \end{aligned}$$

On the other hand, from (29), $\frac{\varepsilon\beta_1}{4M} \geq |\bar{E}_b(\psi) - \bar{E}_b(\bar{\psi})|^{1-\theta} = |\mathcal{E}(\psi) - \mathcal{E}(\bar{\psi})|^{1-\theta}$ hence, (62) holds. ■

4 Weak Solutions

Definition 5 Let $u_0 \in \mathbf{H}$ and $\psi_0 = \phi_0 - m_0 \in H_1^2$, we say that (\mathbf{u}, ψ, z) is a global weak solution of (14)-(19) in $(0, +\infty)$, if

$$\begin{aligned} \mathbf{u} &\in L^2(0, +\infty; \mathbf{V}) \cap L^\infty(0, +\infty; \mathbf{H}), \quad z \in L^2(0, +\infty; H_*^1) \\ \psi &\in L^\infty(0, +\infty; H_1^2), \end{aligned} \quad (30)$$

satisfying the variational formulation

$$\langle \partial_t \mathbf{u}, \bar{\mathbf{u}} \rangle + ((\mathbf{u} \cdot \nabla) \mathbf{u}, \bar{\mathbf{u}}) + \nu(\nabla \mathbf{u}, \nabla \bar{\mathbf{u}}) - \lambda(z \nabla \psi, \bar{\mathbf{u}}) = 0 \quad \forall \bar{\mathbf{u}} \in \mathbf{V}, \quad (31)$$

$$\langle \partial_t \psi, \bar{z} \rangle + (\mathbf{u} \cdot \nabla \psi, \bar{z}) + \gamma(\nabla z, \nabla \bar{z}) = 0, \quad \forall \bar{z} \in H_*^1 \quad (32)$$

$$\varepsilon(\Delta \psi, \Delta \bar{\psi}) + (\overline{G}(\psi), \bar{\psi}) - (z, \bar{\psi}) = 0, \quad \forall \bar{\psi} \in H_1^2, \quad (33)$$

and the initial conditions (19).

Observe that, from (30), (31) and (32), one has $\partial_t \mathbf{u} \in L^{4/5}(0, +\infty; \mathbf{V}')$ and $\partial_t \psi \in L^2(0, +\infty; (H_*^1)')$, hence in particular the initial conditions (19) have sense.

4.1 Energy equality and large-time estimates

In a formal manner, we assume that (\mathbf{u}, ψ, z) is a regular enough solution of (14)-(19). By taking $\bar{\mathbf{u}} = \mathbf{u}$, $\bar{z} = z$ and $\bar{\psi} = \partial_t \psi$ as test function in (31), (32) and (33) respectively,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\mathbf{u}|_2^2 + \nu |\nabla \mathbf{u}|_2^2 - \lambda(z \nabla \psi, \mathbf{u}) &= 0, \\ (\partial_t \psi, z) + (\mathbf{u} \cdot \nabla \psi, z) + \gamma |\nabla z|_2^2 &= 0, \\ \varepsilon \frac{d}{dt} \frac{1}{2} |\Delta \psi|_2^2 + (\overline{G}(\psi), \partial_t \psi) - (z, \partial_t \psi) &= 0. \end{aligned}$$

Adding the first equality plus the second and third ones multiplied by λ , the term $(z, \partial_t \psi)$ cancels, as well as the nonlinear convective term $(\mathbf{u} \cdot \nabla \psi, z)$ with the elastic term $-(z \nabla \psi, \mathbf{u})$, arriving at the following equality

$$\frac{1}{2} \frac{d}{dt} (|\mathbf{u}|_2^2 + \lambda \varepsilon |\Delta \psi|_2^2) + \lambda (\overline{G}(\psi), \partial_t \psi) + \nu |\nabla \mathbf{u}(t)|_2^2 + \lambda \gamma |\nabla z(t)|_2^2 = 0. \quad (34)$$

Since $\frac{\delta \overline{E}_b(\psi)}{\delta \phi} = z$, in particular

$$\frac{d}{dt} \overline{E}_b(\psi(t)) = \left\langle \frac{\delta \overline{E}_b(\psi)}{\delta \psi}, \partial_t \psi \right\rangle = (z, \partial_t \psi) = \varepsilon \frac{1}{2} \frac{d}{dt} |\Delta \psi|_2^2 + (\overline{G}(\psi), \partial_t \psi).$$

We define the total free energy $\overline{E}(\mathbf{u}, \psi) = E_k(\mathbf{u}) + \lambda \overline{E}_b(\psi)$, where $\overline{E}_b(\psi)$ is the bending energy defined in (20) and $E_k(\mathbf{u}) = \frac{1}{2} \int_{\Omega} |\mathbf{u}|^2$ the kinetic energy. Then, equality (34) can be rewritten as the following *energy equality*:

$$\frac{d}{dt} \overline{E}(\mathbf{u}(t), \psi(t)) + \nu |\nabla \mathbf{u}(t)|_2^2 + \lambda \gamma |\nabla z(t)|_2^2 = 0, \quad (35)$$

which shows the dissipative character of the model with respect to the total free energy $\overline{E}(\mathbf{u}, \psi)$. Moreover, assuming the initial estimates (\mathbf{u}_0, ψ_0) in $\mathbf{H} \times H_*^1$, the following uniform “weak” bounds in the infinite time interval $(0, +\infty)$ hold:

$$\mathbf{u} \text{ in } L^\infty(0, +\infty; \mathbf{H}) \cap L^2(0, +\infty; \mathbf{V}), \quad \psi \text{ in } L^\infty(0, +\infty; H_1^2), \quad z \text{ in } L^2(0, +\infty; H^1). \quad (36)$$

4.2 Additional estimates for ψ in H_2^5

The idea is to use previous weak estimates (36) in the ψ -equation (17). Since $\psi \in L^\infty(0, +\infty; H^2)$, in particular $\psi \in L^\infty(0, +\infty; L^\infty)$, then $F'(\psi)$, $F''(\psi)$ and $F'''(\psi)$ are also bounded in $L^\infty(0, +\infty; L^\infty)$. Therefore, we have

$$|\overline{G}(\psi)|_2 \leq C, \quad (37)$$

$$|\nabla \overline{G}(\psi)|_2 \leq C(1 + \|\psi\|_3), \quad (38)$$

$$|\Delta \overline{G}(\psi)|_2 \leq C(1 + \|\psi\|_4). \quad (39)$$

From the ψ -equation (17), by using (23), (37), (38) and (39), we obtain

$$\|\psi\|_4 \leq C(1 + |\overline{G}(\psi)|_2 + |z|_2) \leq C(1 + |z|_2).$$

In particular, from (39) and Poincaré inequality for the mean-value function z ,

$$|\Delta \overline{G}(\psi)|_2 \leq C(1 + |z|_2) \leq C(1 + |\nabla z|_2) \quad (40)$$

On the other hand, again from (17), by using (38) and the interpolation inequality $\|\psi\|_3 \leq \|\psi\|_2^{1/2} \|\psi\|_4^{1/2}$, we deduce $\|\psi\|_5 \leq C(\|z\|_1 + \|\psi\|_2^{1/2} \|\psi\|_4^{1/2})$ and, therefore, since $\|\psi\|_2 \leq C$ and $\|\psi\|_4 \leq C(1 + |z|_2)$, then $\|\psi\|_5 \leq C(\|z\|_1 + 1)$. In particular,

$$\psi \in L_{loc}^2(0, +\infty; H^5). \quad (41)$$

For instance, weak solutions furnished by a limit of Galerkin approximate solutions satisfy the corresponding energy inequality related to (35) (changing in (35) the equality $= 0$ by ≤ 0) and this inequality energy suffices to prove rigorously all previous estimates (36) and (41) for the Galerkin approximations.

Consequently, fixed the initial datum $(\mathbf{u}_0, \psi_0) \in \mathbf{H} \times H_1^2$, by using a Galerkin Method and proceeding in analogous way to Subsection 3.3 of [2], one can prove existence of weak solutions of (14)-(19) in $(0, +\infty)$.

5 Convergence at infinite time.

From the energy inequality (35), we have

$$\overline{E}(\mathbf{u}(t), \psi(t)) - \overline{E}(\mathbf{u}_0, \psi_0) + \int_0^t (\nu |\nabla \mathbf{u}|_2^2 + \lambda \gamma |\nabla z|_2^2) d\tau \leq 0, \quad \forall t \geq 0. \quad (42)$$

Therefore, there exists a number $E_\infty \geq 0$ such that the total energy satisfies

$$\overline{E}(\mathbf{u}(t), \psi(t)) \searrow E_\infty \text{ in } \mathbb{R} \quad \text{as } t \uparrow +\infty. \quad (43)$$

The ω -limit set of a fixed global weak solution, (\mathbf{u}, ψ) , associated to the initial data $(\mathbf{u}_0, \psi_0) \in \mathbf{H} \times H_1^2$, is defined as follows:

$$\begin{aligned} \omega(\mathbf{u}, \psi) &= \{(\mathbf{u}_\infty, \psi_\infty) \in \mathbf{H} \times H_1^2 : \exists \{t_n\} \uparrow +\infty \text{ s.t.} \\ &\quad (\mathbf{u}(t_n), \psi(t_n)) \rightarrow (\mathbf{u}_\infty, \psi_\infty) \text{ weakly in } \mathbf{H} \times H_1^2\}. \end{aligned}$$

Let S be the set of critical points of the energy $\overline{E}(\mathbf{u}, \psi)$:

$$S = \{(0, \psi) : \psi \in H_2^4(\Omega) : \varepsilon \Delta^2 \psi + \overline{G}(\psi) = 0 \text{ a.e in } \Omega\}.$$

Theorem 6 *Assume that $(\mathbf{u}_0, \psi_0) \in \mathbf{H} \times H_1^2$. Fixed $(\mathbf{u}(t), \psi(t), z(t))$ a weak solution of (14)-(19) in $(0, +\infty)$, then, $\omega(\mathbf{u}, \psi)$ is nonempty and $\omega(\mathbf{u}, \psi) \subset S$. Moreover, for any $(0, \overline{\psi}) \in S$ such that $(0, \overline{\psi}) \in \omega(\mathbf{u}, \psi)$, then $\overline{E}_b(\overline{\psi}) = E_\infty$ holds. (In particular, $\mathbf{u}(t) \rightarrow 0$ weakly in $\mathbf{L}^2(\Omega)$ and $\overline{E}_b(\psi(t)) \rightarrow \overline{E}_b(\overline{\psi})$ in \mathbb{R} as $t \uparrow +\infty$).*

Proof. Since $\overline{E}(\mathbf{u}(t_n), \psi(t_n)) \leq \overline{E}(\mathbf{u}_0, \psi_0)$, it exists at least a (non relabeled) subsequence of t_n , such that

$$\mathbf{u}(t_n) \rightarrow \mathbf{u}_\infty \text{ weakly in } \mathbf{H}, \quad \psi(t_n) \rightarrow \psi_\infty \text{ weakly in } H_1^2 \quad (44)$$

for suitable limit functions $(\mathbf{u}_\infty, \psi_\infty) \in \mathbf{H} \times H_1^2$. Let us consider the initial and boundary value problem associated to (14)-(19) on the time interval $[t_n, t_n + 1]$ with initial values $\mathbf{u}(t_n)$ and $\psi(t_n)$. Setting

$$\mathbf{u}_n(t) := \mathbf{u}(t + t_n) \quad \psi_n(t) := \psi(t + t_n) \quad \text{and} \quad z_n(t) := z(\psi(t + t_n)) \quad \text{for a.e. } t \in [0, 1],$$

then, (\mathbf{u}_n, ψ_n) is a weak solution to the problem (14)-(19) on the time interval $[0, 1]$. In particular, from (42),

$$\begin{aligned} &\int_0^1 (\nu |\nabla \mathbf{u}_n|_2^2 + \lambda \gamma |\nabla z_n|_2^2) d\tau \\ &= \int_{t_n}^{t_n+1} (\nu |\nabla \mathbf{u}|_2^2 + \lambda \gamma |\nabla z|_2^2) d\tau \leq \overline{E}(t_n + 1) - \overline{E}(t_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (45)$$

In particular, $\nabla \mathbf{u}_n(t) \rightarrow 0$ strongly in $L^2(0, 1; \mathbf{L}^2)$ and $\nabla z(t_n) \rightarrow 0$ strongly in $L^2(0, 1; L^2)$ whence, by Poincaré inequality,

$$\mathbf{u}_n(t) \rightarrow 0 \text{ strongly in } L^2(0, 1; \mathbf{V}) \quad \text{and} \quad z_n(t) \rightarrow 0 \text{ strongly in } L^2(0, 1; H_*^1). \quad (46)$$

On the other hand, from the large time estimates (36), ψ_n is bounded in $L^\infty(0, 1; H_1^2)$ and \mathbf{u}_n is bounded in $L^2(0, 1; \mathbf{H}^1)$. Then, from the ψ -equation (17), $\partial_t \psi_n$ is bounded in $L^2(0, 1; (\mathbf{H}_*^1)')$, therefore, by using the Aubins-Lions compactness theorem,

$$\psi_n(t) \rightarrow \bar{\psi} \text{ strongly in } C^0([0, 1]; \mathbf{H}_*^1).$$

In particular, $\psi_n(0) \rightarrow \bar{\psi}(0)$ in H_*^1 , hence $\bar{\psi}(0) = \psi_\infty$ is attained because $\psi_n(0) = \psi(t_n) \rightarrow \psi_\infty$ in H_1^2 -weak. Moreover, $\partial_t \psi_n$ converges weakly to $\partial_t \bar{\psi} \equiv 0$ in $L^2(0, 1; (H_*^1)')$, therefore, $\bar{\psi}$ is constant in $[0, 1]$. Consequently,

$$\bar{\psi}(t) = \psi_\infty \quad \text{for all } t \in [0, 1]. \quad (47)$$

Finally, by using convergences (46) and $\bar{G}(\psi_n) \rightarrow \bar{G}(\bar{\psi})$ weakly in $L^2(0, 1; L^2)$, taking limit as $n \rightarrow 0$ in

$$(z_n, \tilde{\psi}) = (\Delta \psi_n, \Delta \tilde{\psi}) + (\bar{G}(\psi_n), \tilde{\psi}), \quad \forall \tilde{\psi} \in H_1^2,$$

we deduce that $(\Delta \psi_n, \Delta \tilde{\psi})$ converges to $(\Delta \bar{\psi}, \Delta \tilde{\psi})$ and also,

$$(\Delta \bar{\psi}, \Delta \tilde{\psi}) + (\bar{G}(\psi_n), \tilde{\psi}) = 0, \quad \forall \tilde{\psi} \in H_1^2, \text{ a.e. } t \in (0, 1),$$

that is $(\Delta \psi_\infty, \Delta \tilde{\psi}) + (\bar{G}(\psi_\infty), \tilde{\psi}) = 0, \forall \tilde{\psi} \in H_1^2$. Therefore, ψ_∞ is a weak solution of the elliptic problem

$$\begin{cases} \Delta^2 \psi + \bar{G}(\psi) = 0 & \text{in } \Omega \\ \partial_n \psi|_{\partial\Omega} = 0, \quad \partial_n \Delta \psi|_{\partial\Omega} = 0, & \int_{\Omega} \psi \, dx = 0. \end{cases}$$

Finally, by using a bootstrap's argument, the regularity $\psi_\infty \in H_2^4$ is deduced and the proof is finished. \blacksquare

Theorem 7 *Under the hypotheses of Theorem 6, there exists a unique limit $\bar{\psi} \in H_2^4$ such that $\psi(t) \rightarrow \bar{\psi}$ in H^2 weakly as $t \uparrow +\infty$, i.e. $\omega(\mathbf{u}, \psi) = \{(0, \bar{\psi})\}$.*

Proof. Let $(0, \bar{\psi}) \in \omega(\mathbf{u}, \psi) \subset S$, i.e, there exists $t_n \uparrow +\infty$ such that $\mathbf{u}(t_n) \rightarrow 0$ weakly in \mathbf{L}^2 and $\psi(t_n) \rightarrow \bar{\psi}$ in H_1^2 weakly.

Without loss of generality, it can be assumed that $\bar{E}(\mathbf{u}(t), \psi(t)) > \bar{E}(0, \bar{\psi}) (= E_\infty)$ for all $t \geq 0$, because otherwise, if it exists some $\tilde{t} > 0$ such that $\bar{E}(\mathbf{u}(\tilde{t}), \psi(\tilde{t})) = \bar{E}(0, \bar{\psi})$, then the energy inequality (35) implies

$$\bar{E}(\mathbf{u}(t), \psi(t)) = \bar{E}(0, \bar{\psi}), \quad |\nabla \mathbf{u}(t)|_2^2 = 0 \quad \text{and} \quad |\nabla z(t)|_2^2 = 0, \quad \text{for each } t \geq \tilde{t}.$$

Therefore, $\mathbf{u}(t) = 0$ and $z(t)$ is constant $\forall t \geq \tilde{t}$. In particular, by using the z -equation, $\partial_t \psi(t) = 0$, and hence $\psi(t) = \bar{\psi}$ for each $t \geq \tilde{t}$. In this situation the convergence of the ψ -trajectory is trivial.

Assuming $\bar{E}(\mathbf{u}(t), \psi(t)) > \bar{E}(0, \bar{\psi}) (= E_\infty)$ for all $t \geq 0$, the proof is divided into three steps.

Step 1: Let $\psi \in H_1^2$ such that $\|\psi(t) - \bar{\psi}\|_1 \leq \beta$, for each $t \geq t_1 \geq 0$, where $\beta > 0$ is the constant appearing in Lemma 9 (of Łojasiewicz-Simon's type), then the following inequalities hold:

$$\frac{d}{dt} \left((\bar{E}(\mathbf{u}(t), \psi(t)) - \bar{E}(0, \bar{\psi}))^\theta \right) + C\theta (|\nabla \mathbf{u}(t)|_2 + |\nabla z(t)|_2) \leq 0, \quad \forall t \in (t_1, \infty) \quad (48)$$

$$\int_{t_1}^{t_2} \|\partial_t \psi\|_{(H_1^1)'} \leq \frac{C}{\theta} (\bar{E}(\mathbf{u}(t_1), \psi(t_1)) - \bar{E}(0, \bar{\psi}))^\theta, \quad \forall t_2 > t_1, \quad (49)$$

where $\theta \in (0, 1/2]$ is the constant appearing in Lemma 4.

Indeed, the energy inequality (35) can be written as

$$\frac{d}{dt} (\bar{E}(\mathbf{u}(t), \psi(t)) - E_\infty) + C (|\nabla \mathbf{u}(t)|_2^2 + |\nabla z(t)|_2^2) \leq 0,$$

hence, in particular, from Poincaré inequality

$$\frac{d}{dt} (\bar{E}(\mathbf{u}(t), \psi(t)) - E_\infty) + C(|\mathbf{u}(t)|_2 + |z(t)|_2) (|\nabla \mathbf{u}(t)|_2 + |\nabla z(t)|_2) \leq 0, \quad \forall t \geq 0.$$

Therefore, by taking the time derivative of the (strictly positive) function

$$H(t) := (\bar{E}(\mathbf{u}(t), \psi(t)) - E_\infty)^\theta > 0,$$

we obtain

$$\frac{dH(t)}{dt} + \theta (\bar{E}(\mathbf{u}(t), \psi(t)) - E_\infty)^{\theta-1} C (|\mathbf{u}(t)|_2 + |z(t)|_2) (|\nabla \mathbf{u}(t)|_2 + |\nabla z(t)|_2) \leq 0, \quad \forall t \geq 0. \quad (50)$$

On the other hand, since the unique critical point of the kinetic energy is $\mathbf{u} = 0$, by taking into account that $|E_k(\mathbf{u}) - E_k(0)| = \frac{1}{2} |\mathbf{u}|_2^2$ and that there exists a constant K such that $|\mathbf{u}(t)|_2 \leq K$, then

$$|E_k(\mathbf{u}(t)) - E_k(0)|^{1-\theta} = \frac{1}{2^{1-\theta}} |\mathbf{u}(t)|_2^{2(1-\theta)} = \frac{1}{2^{1-\theta}} |\mathbf{u}(t)|_2^{1-2\theta} |\mathbf{u}(t)| \leq C |\mathbf{u}(t)|_2 \quad \forall t \geq 0.$$

This estimate together the Łojasiewicz-Simon inequality (given in Lemma 4):

$$|\bar{E}_b(\psi(t)) - \bar{E}_b(\bar{\psi})|^{1-\theta} \leq C (|\mathbf{u}(t)|_2 + |z(t)|_2), \quad \forall t \geq t_1,$$

give

$$\begin{aligned} (\overline{E}(\mathbf{u}(t), \psi(t)) - E_\infty)^{\theta-1} &\geq |E_k(\mathbf{u}(t)) - E_k(0)|^{\theta-1} + |\overline{E}_b(\psi(t)) - \overline{E}_b(\overline{\psi})|^{\theta-1} \\ &\geq C(|\mathbf{u}(t)|_2 + |z(t)|_2)^{-1} \quad \forall t \geq t_1. \end{aligned} \quad (51)$$

Applying (51) in (50),

$$\frac{dH(t)}{dt} + \theta C(|\nabla \mathbf{u}(t)|_2 + |\nabla z(t)|_2) \leq 0, \quad \forall t \geq t_1$$

and (48) is proved.

Integrating (48) into $[t_1, t_2]$ for any $t_2 > t_1$, we have

$$(\overline{E}(\mathbf{u}(t_2), \psi(t_2)) - E_\infty)^\theta + \theta C \int_{t_1}^{t_2} (|\nabla \mathbf{u}(t)|_2 + |\nabla z(t)|_2) dt \leq (\overline{E}(\mathbf{u}(t_1), \psi(t_1)) - E_\infty)^\theta. \quad (52)$$

From the z -equation, $\partial_t \psi = -\mathbf{u} \cdot \nabla \psi + \Delta z$, and by using the weak estimate $\|\psi(t)\|_2 \leq C$, then

$$\|\partial_t \psi\|_{(H_*^1)'} \leq C(|\nabla \mathbf{u}|_2 + |\nabla z|_2)$$

By using this inequality in (52), then (49) is attained.

Step 2: *There exists a sufficiently large n_0 such that $\|\psi(t) - \overline{\psi}\|_1 \leq \beta$ and $|\mathbf{u}(t)|_2 \leq K$ for all $t \geq t_{n_0}$ (β and K given in Lemma 4).*

The bound $|\mathbf{u}(t)|_2 \leq K \quad \forall t > 0$ has been already obtained from weak estimates of \mathbf{u} in $L^\infty(0, +\infty; \mathbf{H})$. We now focus on the bound for $\|\psi(t) - \overline{\psi}\|_1$. Since $\psi(t_n) \rightarrow \overline{\psi}$ in H^1 (strongly) and $\overline{E}(\mathbf{u}(t_n), \psi(t_n)) \rightarrow E_\infty = \overline{E}_b(\overline{\psi})$, then for any $\varepsilon \in (0, \beta)$, there exists an integer $N(\varepsilon)$ such that, for all $n \geq N(\varepsilon)$,

$$\|\psi(t_n) - \overline{\psi}\|_1 \leq \varepsilon \quad \text{and} \quad \frac{1}{\theta} (\overline{E}_b(\mathbf{u}(t_n), \psi(t_n)) - E_\infty)^\theta \leq \varepsilon. \quad (53)$$

For each $n \geq N(\varepsilon)$, we define

$$\bar{t}_n := \sup\{t : t > t_n, \|\psi(s) - \overline{\psi}\|_1 < \beta \quad \forall s \in [t_n, t]\}.$$

It suffices to prove that $\bar{t}_{n_0} = +\infty$ for some n_0 . Assume by contradiction that $t_n < \bar{t}_n < +\infty$ for all n , hence $\|\psi(\bar{t}_n) - \overline{\psi}\|_1 = \beta$ and $\|\psi(t) - \overline{\psi}\|_1 < \beta$ for all $t \in [t_n, \bar{t}_n)$. By applying Step 1 for all $t \in [t_n, \bar{t}_n]$, from (49) and (53) we obtain,

$$\int_{t_n}^{\bar{t}_n} \|\partial_t \psi\|_{(H_*^1)'} \leq C\varepsilon, \quad \forall n \geq N(\varepsilon).$$

Therefore,

$$\|\psi(\bar{t}_n) - \overline{\psi}\|_{(H_*^1)'} \leq \|\psi(t_n) - \overline{\psi}\|_{(H_*^1)'} + \int_{t_n}^{\bar{t}_n} \|\partial_t \psi\|_{(H_*^1)'} \leq (1 + C)\varepsilon,$$

which implies that $\lim_{n \rightarrow +\infty} \|\psi(\bar{t}_n) - \bar{\psi}\|_{(H_*^1)'} = 0$. Since ψ is bounded in $L^\infty(0, +\infty; H_1^2)$, $(\psi(t))_{t \geq t^*}$ is relatively compact in H^1 . Therefore, there exists a subsequence of $\psi(\bar{t}_n)$, which is still denoted as $\psi(\bar{t}_n)$, that converges to $\bar{\psi}$ in H^1 . Hence, $\|\psi(\bar{t}_n) - \bar{\psi}\|_1 < \beta$ for a sufficiently large n , which contradicts the definition of \bar{t}_n .

Step 3: *There exists a unique $\bar{\psi}$ such that $\psi(t) \rightarrow \bar{\psi}$ weakly in H^2 as $t \uparrow +\infty$.*

By using Steps 1 and 2, (49) can be applied, for all $t_1 > t_0 \geq t_{n_0}$, hence

$$\|\psi(t_1) - \psi(t_0)\|_{(H_*^1)'} \leq \int_{t_0}^{t_1} \|\partial_t \psi\|_{(H_*^1)'} \rightarrow 0, \quad \text{as } t_0, t_1 \rightarrow +\infty.$$

Therefore, $(\psi(t))_{t \geq t_{n_0}}$ is a Cauchy sequence in $(H_*^1)'$ as $t \uparrow +\infty$, hence the $(H_*^1)'$ -convergence of the whole trajectory is deduced, i.e. there exists a unique $\bar{\psi} \in (H_*^1)'$ such that $\psi(t) \rightarrow \bar{\psi}$ in $(H_*^1)'$ as $t \uparrow +\infty$. Finally, the weak H^2 -convergence by sequences of $\psi(t)$ proved in Theorem 6, yields $\psi(t) \rightarrow \bar{\psi}$ in H^2 weakly. \blacksquare

6 Higher estimates only for the phase variable

6.1 Global in time strong estimates for ψ

By adding the z -equation (32) tested by $\partial_t \psi \in \mathbf{H}_*^1$, and the ψ -equation (33) by $\Delta \partial_t \psi \in H_1^2$ (see (18)), integrating twice by parts the term $(\bar{G}(\psi) - z, \Delta \partial_t \psi)$, and taking into account that $\nabla \partial_t \psi \cdot \mathbf{n}|_{\partial\Omega} = 0$, $\nabla \bar{G}(\psi) \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $\nabla z \cdot \mathbf{n}|_{\partial\Omega} = 0$, then

$$(\bar{G}(\psi) - z, \Delta \partial_t \psi) = (\Delta \bar{G}(\psi) - \Delta z, \partial_t \psi).$$

Then, the term $(\Delta z, \partial_t \psi)$ cancels, remaining:

$$\varepsilon \frac{1}{2} \frac{d}{dt} |\nabla \Delta \psi|_2^2 + |\partial_t \psi|_2^2 = -(\mathbf{u} \cdot \nabla \psi, \partial_t \psi) - (\Delta \bar{G}(\psi), \partial_t \psi).$$

In particular,

$$\varepsilon \frac{d}{dt} |\nabla \Delta \psi|_2^2 + |\partial_t \psi|_2^2 \leq C(|\mathbf{u} \cdot \nabla \psi|_2^2 + |\Delta \bar{G}(\psi)|_2^2). \quad (54)$$

We bound the convective term as

$$|\mathbf{u} \cdot \nabla \psi|_2^2 \leq |\mathbf{u}|_6^2 |\nabla \psi|_3^2 \leq C \|\mathbf{u}\|_1^2. \quad (55)$$

From (17), (23), (39) and (55), we have that $\|\psi\|_6 \leq C(1 + \|\psi\|_4 + |\partial_t \psi|_2 + \|\mathbf{u}\|_1)$. By using the interpolation $\|\psi\|_4 \leq C\|\psi\|_2^{1/4} \|\psi\|_4^{1/2} \|\psi\|_6^{1/4}$, one has $\|\psi\|_4 \leq C\|\psi\|_6^{1/2}$. Therefore,

$$\|\psi\|_6^2 \leq C(1 + |\partial_t \psi|_2^2 + \|\mathbf{u}\|_1^2). \quad (56)$$

By using (56) in (54),

$$\frac{d}{dt}|\nabla\Delta\psi|_2^2 + C_0\|\psi\|_6^2 \leq C(1 + \|\mathbf{u}\|_1^2 + |\mathbf{u} \cdot \nabla\psi|_2^2 + |\Delta\overline{G}(\psi)|_2^2)$$

and owing to (40) and (55), we have

$$\frac{d}{dt}|\nabla\Delta\psi|_2^2 + C_0\|\psi\|_6^2 \leq C(1 + \|\mathbf{u}\|_1^2 + |\nabla z|_2^2). \quad (57)$$

Since $\|\psi\|_3$ is equivalent to $(|\Delta\psi|_2 + |\nabla\Delta\psi|_2)$ and, taking into account the weak estimates of ψ , $\|\psi\|_3$ is equivalent to $(1 + |\nabla\Delta\psi|_2)$. Then, from (57)

$$\frac{d}{dt}\|\psi\|_3^2 + C_0\|\psi\|_6^2 \leq C(1 + \|\mathbf{u}\|_1^2 + |\nabla z|_2^2). \quad (58)$$

By denoting

$$\Phi(t) := \|\psi\|_3^2, \quad B(t) := \|\mathbf{u}\|_1^2 + |\nabla z|_2^2,$$

then (58) is rewritten as

$$\Phi' + C_0\Phi \leq C(1 + B). \quad (59)$$

Multiplying (59) by e^{C_0t} and integrating in time, we obtain

$$\Phi(t) \leq \Phi(0)e^{-C_0t} + Ce^{-C_0t} \int_0^t e^{C_0s}(1 + B(s)) ds.$$

In particular, $\Phi(t) \leq \Phi(0) + C(1 - e^{-C_0t}) + C \int_0^t B(s) ds$. Since $B(t) \in L^1(0, +\infty)$, we have that $\Phi \in L^\infty([0, +\infty))$. Moreover, integrating (54) and (58) in $[0, t]$, we obtain

$$\psi \in L^\infty(0, +\infty; H_1^3), \quad \psi \in L_{loc}^2(0, +\infty; H_2^6), \quad \partial_t\psi \in L_{loc}^2(0, +\infty; L_*^2). \quad (60)$$

Finally, from (17),

$$z \in L_{loc}^2(0, +\infty; H_1^2).$$

In particular, using this improved estimates for the phase, the phase equation is satisfied point-wisely a.e. $t \in (0, +\infty)$, if the data ψ_0 is sufficiently regular.

Remark 8 *For this model, it has been possible to obtain higher estimates for the phase without improving estimates for the velocity and pressure.*

6.2 Improving the convergence of the phase trajectory

The previous extra regularity obtained for ψ allows to obtain a more regular equilibrium point by using a Łojasiewicz-Simon inequality more demanding than Lemma 4, where the

proximity hypothesis between ψ and $\bar{\psi}$ is imposed in H_1^2 . Fixed the initial data $(\mathbf{u}_0, \psi_0) \in \mathbf{H} \times H_2^3$, the ω -limit set and the set of equilibrium points are defined by:

$$\begin{aligned} \omega(\mathbf{u}, \psi) = \{(\mathbf{u}_\infty, \psi_\infty) \in \mathbf{H} \times H_2^3 : \exists \{t_n\} \uparrow +\infty \text{ s.t.} \\ (\mathbf{u}(t_n), \psi(t_n)) \rightarrow (\mathbf{u}_\infty, \psi_\infty) \text{ weakly in } \mathbf{L}^2 \times H_2^3\}, \end{aligned} \quad (61)$$

$$\mathcal{S} = \{(0, \psi) : \psi \in H_2^4(\Omega) : \varepsilon k \Delta^2 \psi + \bar{G}(\psi) = 0 \text{ a.e in } \Omega\}.$$

Lemma 9 (Lojasiewicz-Simon inequality) *If $(0, \bar{\psi}) \in \mathcal{S}$, there are three positive constants C , β , and $\theta \in (0, 1/2)$ depending on $\bar{\psi}$, such that for all $\psi \in H_2^4$ and $\|\psi - \bar{\psi}\|_2 \leq \beta$, then*

$$|\bar{E}_b(\psi) - \bar{E}_b(\bar{\psi})|^{1-\theta} \leq C \|z\|_2 \quad (62)$$

where $z = z(\psi) := \varepsilon \Delta^2 \psi + \bar{G}(\psi)$.

This result is rather classical. In fact, a similar Lojasiewicz-Simon's lemma is proved in Lemma 5.2 of [12].

In this setting, theorems 6 and 7 can be extended in the following way:

Theorem 10 *The set $\omega(\mathbf{u}, \psi)$ given in (61) is nonempty and $\omega(\mathbf{u}, \psi) \subset \mathcal{S}$. Moreover, for any $(0, \bar{\psi}) \in \mathcal{S}$ such that $(0, \bar{\psi}) \in \omega(\mathbf{u}, \psi)$, then $\bar{E}_b(\bar{\psi}) = E_\infty$ holds.*

Theorem 11 *There exists a unique $\bar{\psi}$ such that $(0, \bar{\psi}) \in \mathcal{S}$ and $\psi(t) \rightarrow \bar{\psi}$ in H_2^3 weakly as $t \uparrow +\infty$, i.e. $\omega(\mathbf{u}, \psi) = \{(0, \bar{\psi})\}$.*

7 Conclusions and Prospects

For the Navier-Stokes-Cahn-Hilliard model introduced in this paper we have proved the convergence of the each trajectory to a single equilibrium point for global weak solutions. Moreover, the regularity for the phase is improved from the energy inequality without the need of more regularity for the velocity and pressure variables, and hence large time or large viscosity constraints are now unnecessary.

Starting of the results obtained in this paper, it seems achievable to obtain (rational) convergence rate estimates of the convergence of trajectories in a similar way to [6]. Finally, it would be interesting to study if local minimizers of the elastic bending energy are the stables, as has been done in [12] for a Navier-Stokes-Allen-Cahn problem modeling vesicle membranes.

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